Homogenization of the Signorini boundary-value problem in a thick plane junction

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Abstract

We consider a mixed boundary-value problem for the Poisson equation in a plane thick junction Ω_ε which is the union of a domain Ω_0 and a large number of $\varepsilon-$periodically situated thin rods. The nonuniform Signorini conditions are given on the vertical sides of the thin rods. The asymptotic analysis of this problem is made as $\varepsilon \to 0$, i.e., when the number of the thin rods infinitely increases and their thickness tends to zero. With the help of the integral identity method we prove the convergence theorem and show that the nonuniform Signorini conditions are transformed (as $\varepsilon \to 0$) in the limiting variational inequalities in the region that is filled up by the thin rods in the limit passage. The existence and uniqueness of the solution to this non-standard limit problem is established. The convergence of the energy integrals is proved as well.

Key words: homogenization; thick junction; Signorini boundary conditions; variational inequalities

MOS subject classification: 35B27, 35R45, 35J20, 74K30

1 Introduction and statement of the problem

Boundary-value problems in thick junctions are mathematical models of widely used engineering and industrial constructions as well as many other physical and biological systems with very distinct characteristic scales. In recent years, a rich collection of new results on asymptotic analysis of boundary-value problems in thick multi-structures is appeared (see [1]–[8]).

In this paper we homogenize the Signorini problem in a thick plane junction of type $2:1:1$ using the integral identity method developed in [9, 10].

A thick junction (or thick multi-structure) of type $k:p:d$ is the union of some domain in $\mathbb{R}^n$, which is called the junction's body, and a large number of ε-periodically situated thin domains along some manifolds on the boundary of the junction's body (see Fig. 1.1). This manifold is called the joint zone. Here ε is a small parameter, which characterizes the distance between neighboring thin domains and their thickness. The type $k:p:d$ of a thick junction refers respectively to the limiting dimensions of the body, the joint zone, and each of the attached thin domains.

This classification of thick junctions was given in [11]–[16] and [9, 10], where rigorous mathematical methods were developed (homogenization, approximation, asymptotic expansions) for analyzing the main boundary-value problems in thick junctions of different types. It was pointed out that qualitative properties of solutions essentially depend on the junction

type and on the conditions given on the boundaries of the attached thin domains. In addition, as it was shown in [17] such problems lose the coercitivity as $\varepsilon \to 0$ and this creates special difficulties in the asymptotic investigation. It should be noted that papers [18, 19] are the first papers in this direction.

For the first time a problem known now as the Signorini problem was considered by Signorini himself in [20]. The sense of the Signorini boundary condition consists in a priori ignorance which of the boundary conditions (Dirichlet or Neumann) are satisfied and where. Many interesting problems in applied mathematics involve the Signorini boundary conditions. Applications arise in groundwater hydrology, in plasticity theory, in crack theory, in optimal control problems, etc. (see [21]). Such of these problems as can be recast as variational inequalities become relatively easy to study (see [21]–[23]). Asymptotic investigations of variational inequalities in perforated domains were made in [24]–[28].

The results of this preprint will be published in [29].

1.1 Statement of the problem

Let a, l be positive numbers, h be a fixed number from the interval $(0,1)$, and N be a large positive integer. Define a small parameter $\varepsilon = \frac{a}{N}$. A model plane thick junction Ω_ε (see Fig. 1.1) consists of the junction's body

$$\Omega_0 = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \quad 0 < x_1 < a, \quad 0 < x_2 < \gamma(x_1) \right\},$$

where $\gamma \in C^1([0,a])$, and a large number of the thin rods

$$G_j(\varepsilon) = \left\{ x : \left| \frac{x_1}{\varepsilon} - (j + \frac{1}{2}) \right| < \frac{h}{2}, \quad x_2 \in [-l, 0] \right\}, \quad j = 0, 1, ..., N-1,$$

i. e. $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon$, where $G_\varepsilon = \bigcup\limits_{j=0}^{N-1} G_j(\varepsilon)$.

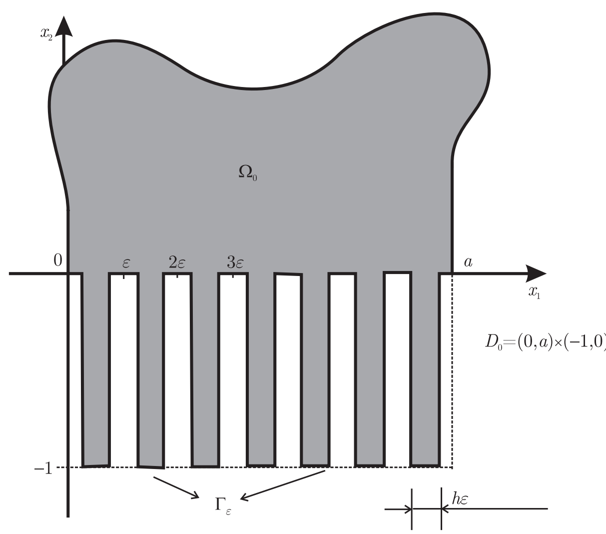

Figure 1: A model plane thick junction Ω_ε of type 2:1:1

The discrete parameter ε characterizes the distance between the rods and their thickness that is equal to εh. Obviously, the thin rods fill out the rectangle $D_0 = (0, a) \times (-l, 0)$ in the limit passage as $N \to +\infty$ $(\varepsilon \to 0)$.

Let us denote the union of vertical sides of the thin rods G_ε by S_ε; the union of bases of the thin rods will be denoted by Γ_ε.

In Ω_ε we consider the following boundary value problem

$$\begin{cases} -\Delta u_\varepsilon(x) = f(x), \quad x \in \Omega_\varepsilon, \\ u_\varepsilon(x) \le g(x), \quad \partial_\nu u_\varepsilon(x) \le \varepsilon d(x), \quad x \in S_\varepsilon, \\ (u_\varepsilon(x) - g(x))\,(\partial_\nu u_\varepsilon(x) - \varepsilon d(x)) = 0, \quad x \in S_\varepsilon, \\ u_\varepsilon(x) = 0, \quad x \in \Gamma_\varepsilon, \qquad \partial_\nu u_\varepsilon(x) = 0, \quad x \in \partial\Omega_\varepsilon \setminus (S_\varepsilon \cup \Gamma_\varepsilon), \end{cases} \tag{1}$$

where $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative, f, g, d are given functions.

We assume that $f \in L^2(\Omega_1)$, where $\overline{\Omega_1} = \overline{\Omega_0} \cup \overline{D_0}$, the function d belongs to the Sobolev space $H^1(D_0)$, and

$$g \in H^1(D_0;\, I_l \cup I_0) = \left\{ v \in H^1(D_0) : \quad v|_{I_l \cup I_0} = 0 \right\},$$

where $I_l = \{x : \quad x_1 \in (0, a), \quad x_2 = -l\}, \quad I_0 = \{x : \quad x_1 \in (0, a), \quad x_2 = 0\}.$

Our goal is to study the asymptotic behavior of the solution u_ε to problem (1) as $\varepsilon \to 0$, i.e., when the number of the thin rods infinitely increases and their thickness tends to zero.

2 Definitions of the weak solution and its existence

In the Sobolev space $H^1(\Omega_\varepsilon; \Gamma_\varepsilon) = \left\{ u \in H^1(\Omega_\varepsilon) : \ u|_{\Gamma_\varepsilon} = 0 \right\}$, we define subset

$$K_\varepsilon = \{\varphi \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon) : \quad \varphi|_{S_\varepsilon} \le g|_{S_\varepsilon} \ \text{a.e. on } S_\varepsilon\},$$

where $\psi|_S$ denotes the trace of a Sobolev function ψ on a curve S. Obviously, K_ε is a closed and convex set for every fixed value of ε.

Let us suppose the existence of a classical solution to problem (1). We can regard that $g = 0$ in Ω_0. Multiplying the equation of problem (1) by the function $(u_\varepsilon - g)$, integrating by parts in Ω_ε and taking into account the boundary conditions for u_ε, we obtain

$$\int_{\Omega_0} \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx + \int_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla (u_\varepsilon - g)\, dx = \int_{\Omega_0} f u_\varepsilon \, dx + \int_{G_\varepsilon} f(u_\varepsilon - g)\, dx + \varepsilon \int_{S_\varepsilon} d(x)(u_\varepsilon - g)\, ds, \tag{2}$$

where $\nabla v \cdot \nabla w = \sum\limits_{j=1}^{n} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_j}$.

Now we take any function $\varphi \in K_\varepsilon$ and multiply the equation of the problem (1) by $(\varphi - g)$. Similar as before we get

$$\int_{\Omega_0} \nabla u_\varepsilon \cdot \nabla \varphi \, dx + \int_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla(\varphi - g)\, dx =$$

$$= \int\limits_{\Omega_0} f\varphi\, dx + \int\limits_{G_\varepsilon} f(\varphi - g)\, dx + \varepsilon \int\limits_{S_\varepsilon} d(x)(\varphi - g)\, ds + \int\limits_{S_\varepsilon} (\partial_\nu u_\varepsilon - \varepsilon d(x))(\varphi - g)\, ds. \quad (3)$$

Since $\partial_\nu u_\varepsilon(x) \le \varepsilon d(x)$ and $\varphi(x) \le g(x)$ a.e. in S_ε,

$$\int\limits_{S_\varepsilon} (\partial_\nu u_\varepsilon - \varepsilon d(x))(\varphi - g)\, ds \ge 0. \quad (4)$$

Taking into account (4), it follows from equality (3) that

$$\int\limits_{\Omega_0} \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \int\limits_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla(\varphi - g)\, dx \ge \int\limits_{\Omega_0} f\varphi\, dx + \int\limits_{G_\varepsilon} f(\varphi - g)\, dx + \varepsilon \int\limits_{S_\varepsilon} d(x)(\varphi - g)\, ds \quad (5)$$

Definition 1. *A function $u_\varepsilon \in K_\varepsilon$ is called a weak solution to problem (1) if it satisfies the integral equality (2) and integral inequality (5) for arbitrary function $\varphi \in K_\varepsilon$.*

Another definition is as follows.

Definition 2. *A function $u_\varepsilon \in K_\varepsilon$ is called a weak solution to problem (1) if it satisfies the integral inequality*

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla(\varphi - u_\varepsilon)\, dx \ge \int\limits_{\Omega_\varepsilon} f(\varphi - u_\varepsilon)\, dx + \varepsilon \int\limits_{S_\varepsilon} d(x)(\varphi - u_\varepsilon)\, ds \quad \forall\, \varphi \in K_\varepsilon. \quad (6)$$

Let us show that these definitions are equivalent. Subtracting equality (2) from inequality (5), we arrive at (6). Setting $\varphi = \begin{cases} 0, & x \in \Omega_0, \\ g, & x \in G_\varepsilon, \end{cases}$ in (6), we have

$$-\int\limits_{\Omega_0} |\nabla u_\varepsilon|^2\, dx + \int\limits_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla(g - u_\varepsilon)\, dx \ge -\int\limits_{\Omega_0} f u_\varepsilon\, dx + \int\limits_{G_\varepsilon} f(g - u_\varepsilon)\, dx + \varepsilon \int\limits_{S_\varepsilon} d(x)(g - u_\varepsilon)\, ds. \quad (7)$$

Putting $\varphi = \begin{cases} 2u_\varepsilon, & x \in \Omega_0, \\ 2u_\varepsilon - g, & x \in G_\varepsilon, \end{cases}$ in (6), we get the reversed inequality

$$\int\limits_{\Omega_0} |\nabla u_\varepsilon|^2\, dx + \int\limits_{G_\varepsilon} \nabla u_\varepsilon \cdot \nabla(u_\varepsilon - g)\, dx \ge \int\limits_{\Omega_0} f u_\varepsilon\, dx + \int\limits_{G_\varepsilon} f(u_\varepsilon - g)\, dx + \varepsilon \int\limits_{S_\varepsilon} d(x)(u_\varepsilon - g)\, ds. \quad (8)$$

This means that (2) holds. Setting $\varphi = \begin{cases} \psi + u_\varepsilon, & x \in \Omega_0, \\ \psi + u_\varepsilon - g, & x \in G_\varepsilon, \end{cases}$ in (6), where ψ is arbitrary function from K_ε, we get (5).

It is well known (see for instance [21]–[23]) that there exists a unique solution of inequality (6) for any fixed value of ε.

3 Auxiliary uniform estimates

To homogenize boundary-value problems in thick multi-structures with the nonhomogeneous Neumann or Fourier conditions on the boundaries of the thin attached domains the method of special integral identities was proposed in [9, 10]. For our problem this identity is as follows (see [10, Lemma 1])

$$\frac{\varepsilon h}{2}\int_{S_\varepsilon} v\,dx_2 = \int_{G_\varepsilon} v\,dx - \varepsilon\int_{G_\varepsilon} Y\left(\frac{x_1}{\varepsilon}\right)\partial_{x_1} v\,dx \qquad \forall\, v \in H^1\big(G_\varepsilon\big), \tag{9}$$

where $Y(\xi) = -\xi + [\xi] + \frac{1}{2}$, $[\xi]$ is the integral part of ξ.

Using (9) and taking into account that $\max_{\mathbb{R}} |Y| \le 1$, we get

$$\|v\|_{L^2(S_\varepsilon)} \le C_1 \varepsilon^{-\frac{1}{2}} \|v\|_{H^1(G_\varepsilon)} \quad \forall\, v \in H^1(G_\varepsilon). \tag{10}$$

Remark 1. *Here and in what follows all constants $\{C_i\}$ and $\{c_i\}$ in inequalities are independent of the parameter ε.*

Also it will be very important for our research the following uniform estimates.

Lemma 1 ([14])**.** *The usual norm $\|u\|_{H^1(\Omega_\varepsilon)} = \left(\int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2)\,dx\right)^{\frac{1}{2}}$ in $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ and a norm $\|\cdot\|_\varepsilon$, which is generated by scalar product*

$$(u, v)_\varepsilon = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v\,dx, \quad u, v \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon),$$

are uniformly equivalent, i.e., there exist constants $C_1 > 0$ and $\varepsilon_0 > 0$ such for all $\varepsilon \in (0, \varepsilon_0)$ and for all $u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ the following estimates

$$\|u\|_\varepsilon \le \|u\|_{H^1(\Omega_\varepsilon)} \le C_1 \|u\|_\varepsilon \tag{11}$$

hold.

Remark 2. *In fact, in Lemma 1 the following Friedrich inequality*

$$\|u\|_{L^2(\Omega_\varepsilon)} \le C_2 \|\nabla u\|_{L^2(\Omega_\varepsilon)} \quad \forall\, u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \tag{12}$$

was proved.

Using the Cauchy–Bunyakovsky integral inequality and Cauchy's inequality with $\delta > 0$ ($2ab \le \delta a^2 + \delta^{-1} b^2$ for any positive numbers a and b), with the help of (10) and (12) we deduce from (2) that

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2\,dx \le c_0(\delta_1 + \delta_2 + \delta_3)\,\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \\ + c_1(1 + \delta_1^{-1})\,\|g\|^2_{H^1(D_0)} + c_2(1 + \delta_2^{-1})\,\|f\|^2_{L^2(\Omega_\varepsilon)} + c_3(1 + \delta_3^{-1})\,\|d\|^2_{H^1(D_0)}. \tag{13}$$

Choosing $\delta_1, \delta_2, \delta_3$ so that $c_1(\delta_1 + \delta_2 + \delta_3) < \frac{1}{2}$, we have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \le c_4 \left(\|f\|^2_{L^2(\Omega_1)} + \|g\|^2_{H^1(D_0)} + \|d\|^2_{H^1(D_0)}\right). \tag{14}$$

By virtue of (11) we obtain from (14) the following uniform estimate

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C_3. \tag{15}$$

4 Convergence theorem

In the sequel, $\widetilde{u}$ denotes the zero extension of a function u defined on G_ε to the rectangle D_0, which is filled up by the thin rods in the limit passage as $\varepsilon \to 0$, namely:

$$\widetilde{u}(x) = \begin{cases} u(x), & x \in G_\varepsilon, \\ 0, & x \in D_0 \setminus G_\varepsilon. \end{cases}$$

If $u \in H^1(G_\varepsilon, \Gamma_\varepsilon)$, then due to the rectilinearity of the boundaries of the thin rods $\widetilde{u}$ belongs to the anisotropic Sobolev space

$$H^{0,1}(D_0; I_l) = \big\{ v \in L^2(D_0) : \exists \ \text{ weak derivative } \ \partial_{x_2} v \in L^2(D_0) \ \text{ and } \ v|_{I_l} = 0 \big\} \tag{16}$$

and $\partial_{x_2}\widetilde{u} = \widetilde{\partial_{x_2} u}$ a.e. in D_0.

Theorem 1. *The solution u_ε to problem (1) satisfies the relations*

$$\left.\begin{array}{rcll} u_\varepsilon|_{\Omega_0} & \stackrel{w}{\longrightarrow} & u_0^+ & \textit{weakly in } \ H^1(\Omega_0), \\ \widetilde{u}_\varepsilon & \stackrel{w}{\longrightarrow} & h\, u_0^- & \textit{weakly in } \ W^{0,1}(D_0, I_l), \\ \widetilde{\partial_{x_1} u_\varepsilon} & \stackrel{w}{\longrightarrow} & 0 & \textit{weakly in } \ L^2(D_0) \end{array}\right\} \quad \textit{as} \quad \varepsilon \to 0, \tag{17}$$

and the function $u_0(x) = \begin{cases} u_0^+, & x \in \Omega_0, \\ u_0^-, & x \in D_0, \end{cases}$ *is a unique solution of the following problem*

$$\begin{cases} -\Delta u_0^+(x) & = f(x), & x \in \Omega_0, \\ -h\, \partial^2_{x_2x_2} u_0^-(x) & \le h\, f(x) + 2\, d(x), & x \in D_0, \\ u_0^-(x) & \le g(x), & x \in D_0, \\ (u_0^-(x) - g(x))\, (h\, \partial^2_{x_2x_2} u_0^-(x) & +hf(x) + 2d(x)) = 0, & x \in D_0, \\ \partial_\nu u_0^+(x) & = 0, & x \in \partial\Omega_0 \setminus I_0, \\ u_0^-(x_1, -l) & = 0, & x_1 \in [0, a], \\ u_0^+(x_1, 0) & = u_0^-(x_1, 0), & x_1 \in [0, a], \\ \partial_{x_2} u_0^+(x_1, 0) & = h\, \partial_{x_2} u_0^-(x_1, 0), & x_1 \in [0, a], \end{cases} \tag{18}$$

which is called the homogenized problem for (1).

Furthermore, the following energy convergence holds

$$\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = E_0(u_0),$$

where

$$E_\varepsilon(u_\varepsilon) = \int\limits_{\Omega_0} |\nabla u_\varepsilon|^2\, dx, \quad E_0(u_0) = \int\limits_{\Omega_0} |\nabla u_0^+|^2\, dx + h \int\limits_{D_0} |\partial_{x_2} u_0^-|^2\, dx.$$

Before the proof of Theorem 1 we investigate the homogenized problem (18).

4.1 The solvability of the homogenized problem

We see that the homogenized problem (18) is a non-standard boundary-value problem that consists of the Poisson equation in the junction body Ω_0, the variational inequalities in D_0 and the transmission conditions in the joint zone I_0. Therefore, at first we give the definition of the weak solution to this problem and then with the help of general approach in the theory of variational inequalities we prove the existence and uniqueness.

Let us introduce partly anisotropic Sobolev space

$$\mathcal{H}(\Omega_1, D_0; I_l) = \left\{ u \in L^2(\Omega_1) : \ \ \exists\, \partial_{x_2} u \in L^2(\Omega_1), \quad u|_{\Omega_0} \in H^1(\Omega_0), \quad u|_{D_0} \in H^{0,1}(D_0; I_l) \right\},$$

where $H^{0,1}(D_0; I_l)$ is defined in (16). It follows from properties of anisotropic Sobolev spaces (see [30]) that the traces of the restrictions $u^+ := u|_{\Omega_0}$ and $u^- := u|_{D_0}$ on I_0 are equal. In addition, since traces of functions from $\mathcal{H}(\Omega_1, D_0; I_l)$ vanish on I_l, there exists a constant C_0 such that

$$\int_{\Omega_1} u^2\, dx \le C_0 \Big(\int_{\Omega_0} |\nabla u^+|^2\, dx + \int_{D_0} |\partial_{x_2} u^-|^2\, dx \Big) \quad \forall\, u \in \mathcal{H}(\Omega_1, D_0; I_l).$$

In $\mathcal{H}(\Omega_1, D_0; I_l)$ we introduce a norm $\|\cdot\|_{\mathcal{H}}$, which is generated by a scalar product

$$(u, v)_{\mathcal{H}} = \int_{\Omega_0} \nabla u^+ \cdot \nabla v^+\, dx + h \int_{D_0} \partial_{x_2} u^- \, \partial_{x_2} v^-\, dx, \quad u,\ v \in \mathcal{H}(\Omega_1, D_0; I_l).$$

We now define subset $K_0 = \{\varphi \in \mathcal{H}(\Omega_1, D_0; I_l) : \ \varphi^- \le g \ \text{ a.e. in } D_0\}$ in $\mathcal{H}(\Omega_1, D_0; I_l)$. Obviously, K_0 is a closed and convex set.

Definition 3. *A function $u_0 \in K_0$ is called a weak solution of problem (18) if it satisfies the integral equality*

$$\int_{\Omega_0} \nabla u_0^+ \cdot \nabla u_0^+\, dx + h \int_{D_0} \partial_{x_2} u_0^- \partial_{x_2}(u_0^- - g)\, dx = \int_{\Omega_0} f u_0^+\, dx + h \int_{D_0} f\,(u_0^- - g)\, dx + 2 \int_{D_0} d\,(u_0^- - g)\, dx, \tag{19}$$

and the integral inequality

$$\int_{\Omega_0} \nabla u_0^+ \cdot \nabla \varphi\, dx + h \int_{D_0} \partial_{x_2} u_0^- \partial_{x_2}(\varphi - g)\, dx \ge \int_{\Omega_0} f \varphi\, dx + h \int_{D_0} f\,(\varphi - g)\, dx + 2 \int_{D_0} d\,(\varphi - g)\, dx \tag{20}$$

for arbitrary function $\varphi \in K_0$.

If there exists a classical solution to the homogenized problem (18), then relations (19) and (20) can be obtained by the same way as relations (2) and (5) in Definition 1. Similarly as we proved the equivalence of Definitions 1 and 2, we can show the equivalence of Definition 3 to the following definition.

Definition 4. *A function $u_0 \in K_0$ is called a weak solution to problem (18) if it satisfies the integral inequality*

$$\int_{\Omega_0} \nabla u_0^+ \cdot \nabla(\varphi - u_0^+)\, dx + h \int_{D_0} \partial_{x_2} u_0^- \, \partial_{x_2}(\varphi - u_0^-)\, dx \ge$$

$$\ge \int_{\Omega_0} f(\varphi - u_0^+)\, dx + h \int_{D_0} f\,(\varphi - u_0^-)\, dx + 2 \int_{D_0} d\,(\varphi - u_0^-)\, dx \tag{21}$$

for arbitrary function $\varphi \in K_0$.

We now give the third definition of the weak solution to problem (18).

Definition 5. *A function $u_0 \in K_0$ is called a weak solution to problem (18) if it satisfies the integral inequality*

$$\int_{\Omega_0} \nabla\varphi \cdot \nabla(\varphi - u_0^+)\, dx + h \int_{D_0} \partial_{x_2}\varphi\, \partial_{x_2}(\varphi - u_0^-)\, dx \geq$$

$$\geq \int_{\Omega_0} f(\varphi - u_0^+)\, dx + h \int_{D_0} f\,(\varphi - u_0^-)\, dx + 2\int_{D_0} d\,(\varphi - u_0^-)\, dx \tag{22}$$

for arbitrary function $\varphi \in K_0$.

Let us prove that Definition 4 and Definition 5 are equivalent. Adding the inequality

$$\int_{\Omega_0} \nabla(\varphi - u_0^+) \cdot \nabla(\varphi - u_0^+)\, dx + h \int_{D_0} \partial_{x_2}(\varphi - u_0^-)\, \partial_{x_2}(\varphi - u_0^-) \geq 0 \quad (\varphi \in K_0)$$

to inequality (21), we get (22). Now we take any $\psi \in K_0$. Setting $\varphi = u_0 + t(\psi - u_0) \in K_0$ (for any $t \in [0,1]$) in inequality (22), we obtain

$$\int_{\Omega_0} \nabla(u_0^+ + t(\psi - u_0^+)) \cdot \nabla(\psi - u_0^+)\, dx + h \int_{D_0} \partial_{x_2}(u_0^- + t(\psi - u_0^-))\, \partial_{x_2}(\psi - u_0^-)\, dx \geq$$

$$\geq \int_{\Omega_0} f(\psi - u_0^+)\, dx + h \int_{D_0} f\,(\psi - u_0^-)\, dx + 2\int_{D_0} d\,(\psi - u_0^-)\, dx. \tag{23}$$

Passing to the limit in (23) as $t \to 0$, we arrive at (21). Thus, all Definitions 3, 4 and 5 are equivalent.

We can re-write inequality (21) in the following form

$$(u, \varphi - u)_{\mathcal{H}} \geq \langle F, \varphi - u \rangle \quad \forall\, \varphi \in K_0, \tag{24}$$

where F is a linear continuous functional on $\mathcal{H}(\Omega_1, D_0; I_l)$ defined by the formulae

$$\langle F, w \rangle = \int_{\Omega_0} f\, w^+\, dx + h \int_{D_0} f\, w^-\, dx + 2\int_{D_0} d\, w^-\, dx \quad \text{for all } w \in \mathcal{H}(\Omega_1, D_0; I_l).$$

Using the theory of variational inequalities in Hilbert spaces (see [21, Sec. 2]), we can state that there exists a unique solution of the inequality (24) and consequently of the homogenized problem (18).

4.2 The proof of Theorem 1

1. From (15) it follows that $\|u_\varepsilon\|_{H^1(\Omega_0)} \leq C_3$, $\|\widetilde{u_\varepsilon}\|_{L^2(D_0)} \leq C_3$ and $\|\widetilde{\partial_{x_i} u_\varepsilon}\|_{L^2(D_0)} \leq C_3$, $i = 1, 2$. Therefore we can choose a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$ (again denoted by ε), such that

$$\left.\begin{array}{rcll} u_\varepsilon|_{\Omega_0} & \stackrel{w}{\longrightarrow} & u_0^+ & \text{weakly in } H^1(\Omega_0), \\ \widetilde{u}_\varepsilon & \stackrel{w}{\longrightarrow} & h\, u_0^- & \text{weakly in } L^2(D_0), \\ \widetilde{\partial_{x_i} u_\varepsilon} & \stackrel{w}{\longrightarrow} & \gamma_i & \text{weakly in } L^2(D_0), \quad i = 1, 2, \end{array}\right\} \quad \text{as} \quad \varepsilon \to 0, \tag{25}$$

where u_0^+, u_0^-, γ_1, γ_2 are some functions which will be determined later.

At first we determine γ_2. Take any function $\psi \in C_0^\infty(D_0)$ and perform the following calculations

$$\int\limits_{D_0} \widetilde{\partial_{x_2} u_\varepsilon}\, \psi\, dx = \int\limits_{D_0} \partial_{x_2} \widetilde{u_\varepsilon}\, \psi\, dx = \int\limits_{G_\varepsilon} \partial_{x_2} u_\varepsilon\, \psi\, dx = -\int\limits_{G_\varepsilon} u_\varepsilon\, \partial_{x_2}\psi\, dx = -\int\limits_{D_0} \widetilde{u_\varepsilon}\, \partial_{x_2}\psi\, dx.$$

Passing to the limit in this identity, as $\varepsilon \to 0$, we obtain

$$\int_{D_0} \gamma_2\, \psi\, dx = -h \int_{D_0} u_0^-\, \partial_{x_2}\psi\, dx, \qquad \forall \psi \in C_0^\infty(D_0), \tag{26}$$

whence it follows that there exists a weak derivative $\partial_{x_2} u_0^-$ and $\gamma_2 = h\, \partial_{x_2} u_0^-$ a. e. in D_0.

Now let us find γ_1. Consider the function

$$\Phi(x) = \begin{cases} 0, & x \in \Omega_0, \\ \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right)\psi + g, & x \in G_\varepsilon, \end{cases} \qquad \forall \psi \in C_0^\infty(D_0), \quad \psi \geq 0,$$

where $Y_1(\xi) = -\xi + [\xi]$. It is easy to see that $\Phi \in K_\varepsilon$ and

$$\nabla(\Phi - g) = \left(-\psi + \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right)\partial_{x_1}\psi,\ \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right)\partial_{x_2}\psi\right), \qquad x \in G_\varepsilon.$$

Substituting the function $\Phi - g$ into the integral inequality (5) for solution u_ε, we get

$$\int\limits_{G_\varepsilon} \left(-\partial_{x_1} u_\varepsilon\, \psi + \varepsilon Y_1\left(\frac{x_1}{\varepsilon}\right)\nabla u_\varepsilon \cdot \nabla\psi\right)\, dx \geq \varepsilon \int\limits_{G_\varepsilon} Y_1\left(\frac{x_1}{\varepsilon}\right) f\, \psi\, dx - \frac{\varepsilon^2(1 \pm h)}{2} \int\limits_{S_\varepsilon^\pm} d\, \psi\, dx_2,$$

where the sings " $+$ " or " $-$ " in $S_\varepsilon^\pm$ indicate the union of the right or left sides of the thin rods respectively. With the help of (10) and (15) we deduce from previous inequality the estimate

$$\begin{aligned} \left|\int_{D_0} \widetilde{\partial_{x_1} u_\varepsilon}\, \psi\, dx\right| &\leq \varepsilon \left(\int_{G_\varepsilon} \left|Y_1\left(\frac{x_1}{\varepsilon}\right)(\nabla u_\varepsilon \cdot \nabla\psi - f\,\psi)\right| dx + \frac{\varepsilon(1+h)}{2} \int_{S_\varepsilon} |d\,\psi|\, dx_2\right) \leq \\ &\leq \varepsilon c_1 \left(\|\nabla u_\varepsilon\|_{L^2(G_\varepsilon)} \|\nabla\psi\|_{L^2(G_\varepsilon)} + \|f\|_{L^2(G_\varepsilon)} \|\psi\|_{L^2(G_\varepsilon)} + \varepsilon \|d\|_{L^2(S_\varepsilon)} \|\psi\|_{L^2(S_\varepsilon)}\right) \leq \\ &\leq \varepsilon c_1 \left(\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \|\psi\|_{H^1(D_0)} + \|f\|_{L^2(\Omega_1)} \|\psi\|_{L^2(D_0)} + \|d\|_{H^1(D_0)} \|\psi\|_{H^1(D_0)}\right) \leq \varepsilon c_2, \end{aligned}$$

from which, passing to the limit as $\varepsilon \to 0$, we get $\int_{D_0} \gamma_1 \psi\, dx = 0$ for all $\psi \in C_0^\infty(D_0)$, $\psi \geq 0$. This means that $\gamma_1 = 0$ a. e. in D_0.

2. Let us show that the traces of the functions u_0^+ and u_0^- on I_0 are equal. By virtue of the compactness of the trace operator and the first relation in (25), we have

$$u_\varepsilon(x_1, 0) \xrightarrow{s} u_0^+(x_1, 0) \quad \text{in } L^2(0, a) \quad \text{as} \quad \varepsilon \to 0. \tag{27}$$

Consider the following equality

$$\widetilde{u_\varepsilon}(x_1, 0) = \chi_h\left(\frac{x_1}{\varepsilon}\right) u_\varepsilon(x_1, 0) \quad \text{for a. e. } x_1 \in (0, a), \tag{28}$$

where $\chi_h(\xi),\ \xi \in \mathbb{R}$, is the 1-periodic function, defined on the segment $[0,1]$ as follows:

$$\chi_h(\xi) = \begin{cases} 1, & |\xi - \frac{1}{2}| \le \frac{h}{2}, \\ 0, & \frac{h}{2} < |\xi - \frac{1}{2}| \le 1. \end{cases}$$

It is known that $\chi_h\left(\frac{x_1}{\varepsilon}\right) \xrightarrow{w} h$ weakly in $L^2(0,1)$ as $\varepsilon \to 0$. Using this fact and (27), we obtain that the right-hand side in (28) converges to $h\, u_0^+(x_1,0)$ weakly in $L^2(0,a)$. On the other hand,

$$\int\limits_0^a \widetilde{u_\varepsilon}(x_1,0)\psi(x_1)\,dx_1 = \frac{1}{l}\int\limits_{D_0} \widetilde{u_\varepsilon}(x)\psi(x_1)\,dx + \frac{1}{l}\int\limits_{D_0}(x_2+l)\widetilde{\partial_{x_2}u_\varepsilon}\cdot\psi(x_1)\,dx \quad \forall \psi \in C_0^\infty(0,a). \quad (29)$$

Passing to the limit in (29) as $\varepsilon \to 0$ and taking (26) into account, we have

$$h\int\limits_0^a u_0^+(\cdot,0)\psi(x_1)\,dx_1 = \frac{h}{l}\int\limits_{D_0} u_0^-\psi(x_1)\,dx + \frac{h}{l}\int\limits_{D_0}(x_2+l)\,\partial_{x_2}u_0^-\,\psi(x_1)\,dx \quad \forall\, \psi \in C_0^\infty(0,a),$$

whence it appears

$$\int_0^a u_0^+(\cdot,0)\psi(x_1)\,dx_1 = \int_0^a u_0^-(\cdot,0)\psi(x_1)\,dx_1 \quad \forall\, \psi \in C_0^\infty(0,a),$$

i.e., $u_0^+(x_1,0) = u_0^-(x_1,0)$ for a.e. $x_1 \in (0,a)$.

Similarly we can prove that the trace $u_0^-|_{I_l}$ is equal to zero.

Thus, the results obtained above mean that the function

$$u_0(x) = \begin{cases} u_0^+, & x \in \Omega_0, \\ u_0^-, & x \in D_0, \end{cases}$$

belongs to the space $\mathcal{H}(\Omega_1, D_0; I_l)$.

3. Let us add the inequality

$$\int_{\Omega_0} \nabla(\varphi - u_\varepsilon)\cdot\nabla(\varphi - u_\varepsilon)\,dx + \int_{G_\varepsilon} \partial_{x_2}(\varphi - u_\varepsilon)\,\partial_{x_2}(\varphi - u_\varepsilon)\,dx + \int_{G_\varepsilon} \partial_{x_1}u_\varepsilon\,\partial_{x_1}u_\varepsilon\,dx \ge 0,$$

where φ is arbitrary function from $C^1(\overline{\Omega_1})$ such that $\varphi|_{I_l} = 0$ and $\varphi \le g$ in D_0 (obviously $\varphi|_{\Omega_\varepsilon} \in K_\varepsilon$), to inequality (6). We get

$$\int_{\Omega_0} \nabla\varphi\cdot\nabla(\varphi - u_\varepsilon)\,dx + \int_{G_\varepsilon} \partial_{x_1}u_\varepsilon\,\partial_{x_1}\varphi\,dx + \int_{G_\varepsilon} \partial_{x_2}\varphi\,\partial_{x_2}(\varphi - u_\varepsilon)\,dx \ge$$

$$\ge \int_{\Omega_\varepsilon} f(\varphi - u_\varepsilon)\,dx + \varepsilon\int_{S_\varepsilon} d(x)(\varphi - u_\varepsilon)\,ds,$$

which we can re-write with the help of (9) in the following view

$$\int\limits_{\Omega_0} \nabla\varphi\cdot\nabla(\varphi-u_\varepsilon)\,dx+\int\limits_{D_0}\widetilde{\partial_{x_1}u_\varepsilon}\,\partial_{x_1}\varphi\,dx+\int\limits_{D_0}\chi_h\left(\frac{x_1}{\varepsilon}\right)\partial_{x_2}\varphi\,\partial_{x_2}\varphi\,dx-\int\limits_{D_0}\partial_{x_2}\varphi\,\widetilde{\partial_{x_2}u_\varepsilon}\,dx\ \geq$$

$$\geq\int\limits_{\Omega_0} f\,(\varphi-u_\varepsilon)\,dx+\int\limits_{D_0}\chi_h\left(\frac{x_1}{\varepsilon}\right)f\,\varphi\,dx-\int\limits_{D_0}f\widetilde{u_\varepsilon}\,dx+$$

$$+\frac{2}{h}\int\limits_{D_0}\chi_h\left(\frac{x_1}{\varepsilon}\right)d\,\varphi\,dx-\frac{2}{h}\int\limits_{D_0}d\,\widetilde{u_\varepsilon}\,dx-\frac{2\varepsilon}{h}\int_{G_\varepsilon}Y\left(\frac{x_1}{\varepsilon}\right)\partial_{x_1}(d(\varphi-u_\varepsilon))\,dx. \quad (30)$$

Passing to the limit in (30) as $\varepsilon\to0$ and taking into account results obtained above, we obtain the following integral inequality

$$\int\limits_{\Omega_0}\nabla\varphi\cdot\nabla(\varphi-u_0^+)\,dx+h\int\limits_{D_0}\partial_{x_2}\varphi\,\partial_{x_2}(\varphi-u_0^-)\,dx\geq$$

$$\geq\int\limits_{\Omega_0}f(\varphi-u_0^+)\,dx+h\int\limits_{D_0}f\,(\varphi-u_0^-)\,dx+2\int\limits_{D_0}d\,(\varphi-u_0^-)\,dx \quad (31)$$

for any function $\varphi\in K_1=\left\{\varphi\in C^1(\overline{\Omega_1}):\ \varphi\,|_{I_l}=0,\ \varphi\leq g \text{ in } D_0\right\}$.

Since the set K_1 is dense in K_0, the integral inequality (31) holds for any function $\varphi\in K_0$. This means that the function u_0 is the unique solution of inequality (21) (see Definition 4) and also it is the weak solution to the homogenized problem (18).

Due to the uniqueness of the solution to problem (18), the above argumentations are true for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. Thus the limits (17) hold.

4. ¿From equalities (2) and (19) it follows that

$$E_\varepsilon(u_\varepsilon)=\int\limits_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx=\int\limits_{G_\varepsilon}\nabla u_\varepsilon\cdot\nabla g\,dx+\int\limits_{\Omega_0}fu_\varepsilon\,dx+\int\limits_{G_\varepsilon}f(u_\varepsilon-g)dx+\varepsilon\int\limits_{S_\varepsilon}d(x)(u_\varepsilon-g)\,ds, \quad (32)$$

$$E_0(u_0)=\int\limits_{\Omega_0}|\nabla u_0^+|^2\,dx+h\int\limits_{D_0}|\partial_{x_2}u_0^-|^2\,dx=h\int\limits_{D_0}\partial_{x_2}u_0^-\,\partial_{x_2}g\,dx+$$

$$+\int\limits_{\Omega_0}fu_0^+\,dx+h\int\limits_{D_0}f\,(u_0^--g)\,dx+2\int\limits_{D_0}d\,(u_0^--g)\,dx. \quad (33)$$

Passing to the limit in (32) similarly as we made this in (30) and taking into account (33), we obtain $\lim_{\varepsilon\to0}E_\varepsilon(u_\varepsilon)=E_0(u_0)$. The theorem is proved.

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